TWO DEVELOPMENTS.

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It is desired to call attention here to two developments, whose statement and discussion the writer has no where met, except for $n=2$, and then not from the point of view to be suggested here. Yet it is quite possible that they may be found elsewhere.*

I. Formulas.

If $u=f(x_1, x_2, \ldots\ldots x_n)$ is a function of n variables, Δu will be used to denote the total increment in the function due to a change in all the variables, while $\Delta_{x_1, x_2 \ldots\ldots x_r} u$ will denote an increment due to a change in r variables. The two developments are as follows:

1. $u + \Delta u = u + \Delta_{x_1} u + \Delta_{x_2} u + \ldots\ldots \Delta_{x_n} u$

$$+ \Delta_{x_1}\Delta_{x_2} u + \Delta_{x_1}\Delta_{x_3} u + \ldots\ldots \Delta_{x_1}\Delta_{x_2}\ldots\ldots \Delta_{x_n} u$$

$$= u + \sum_{r=1}^{r=n}{}^r \sum_{\alpha_1=1}^{\alpha_1=n-r+1}{}^{\alpha_1} \sum_{\alpha_2=\alpha_1+1}^{\alpha_2=n-r+2}{}^{\alpha_2}\ldots\ldots$$

$$\sum_{\alpha_{r-1}=\alpha_{r-2}+1}^{\alpha_{r-1}=n-1}{}^{\alpha_{r-1}} \sum_{\alpha_r=\alpha_{r-1}+1}^{\alpha_r=n}{}^{\alpha_r}\Delta_{x_{a_1}}\Delta_{x_{a_2}}\ldots\ldots\Delta_{x_{a_r}} u.$$

Or in determinant notation,

$$\begin{vmatrix} \Delta & 1 \\ -1 & 1 \end{vmatrix} u = \begin{vmatrix} \Delta_{x_1} & 1 & 1\ldots\ldots\ldots 1 & 1 \\ -1 & \Delta_{x_2} & 1\ldots\ldots\ldots 1 & 1 \\ -1 & -1 & \cdot & \vdots & \vdots \\ \ldots\ldots\ldots & & \cdot & \Delta_{x_n} & 1 \\ -1 & -1 & \ldots\ldots -1 & 1 \end{vmatrix} u$$

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*Since the above was read Professor Fiske of Columbia, has informed the writer that "Dr. McClintock called attention to the fact, that the first result is contained in a general formula which he gave in Vol. II. page 116 of the *American Journal of Mathematics*. His formula (77) reduces in a special case to

$$\phi(E) = \phi(E_{x_1} E_{x_2} \ldots\ldots E_{x_n}) \text{ where } E = 1 + \Delta.$$

His operation S reduces to E when

$$\Psi = 1."$$

$$=(1+\Delta_{x_1})(1+\Delta_{x_2})\ldots\ldots(1+\Delta_{x_n}u)=\overset{r=n}{\underset{r=1}{\pi}}r(1+\Delta_{x_r})u,$$

so that as operators $(1+\Delta)=\overset{r=n}{\underset{r=1}{\pi}}r(1+\Delta_{x_r})$, or the operator $1+\Delta$ is developable into the product of n operators.

2. $u-\int du=u-\int d_{x_1}u-\int d_{x_2}u\ldots\ldots-\int d_{x_n}u+\int\int d_{x_1}d_{x_2}u+\int\int d_{x_1}d_{x_3}u+$

$$\ldots\ldots+(-1)^n\int\int\ldots\ldots\int d_{x_1}d_{x_2}\ldots\ldots d_{x_n}u$$

$$=u+\overset{r=n}{\underset{r=1}{\sum}}r\ \overset{\alpha_1=n-r+1}{\underset{\alpha_1=1}{\sum}}\alpha_1\ \overset{\alpha_2=n-r+2}{\underset{\alpha_2=\alpha_1+1}{\sum}}\alpha_2\ldots\ldots$$

$$\overset{\alpha_r=n}{\underset{\alpha_r=\alpha_{r-1}+1}{\sum}}\alpha_r(-1)^r\int\int\ldots\ldots\int d_{x_{a_1}}d_{x_{a_2}}\ldots\ldots d_{x_{a_r}}u.$$

Or in determinant notation,

$$\begin{vmatrix} -\int d & 1 \\ -1 & 1 \end{vmatrix}u==\begin{vmatrix} -\int d_{x_1} & 1 & 1\ldots\ldots 1 & 1 \\ -1 & -\int d_{x_2} & 1\ldots\ldots 1 & 1 \\ -1 & -1 & & \vdots & \vdots \\ \ldots\ldots\ldots\ldots & & -\int d_{x_n} & 1 \\ -1 & -1\ldots\ldots & & -1 & 1 \end{vmatrix}u$$

$$=(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=\overset{r=n}{\underset{r=1}{\pi}}r(1-\int d_{x_r})u=0,$$

so that as operators $(1-\int d)=\overset{r=n}{\underset{r=1}{\pi}}r(1-\int d_{x_r})$, or the operator $1-\int d$ is developable into the product of n operators.

II. Proofs.

1. Let $u=f(x_1)$, then $\Delta u=\Delta_{x_1}u$, and $u+\Delta u=u+\Delta_{x_1}u=(1+\Delta_{x_1})u$.

Let $u=f(x_1 x_2)$, then by the preceding, $u+\Delta_{x_2}u=(1+\Delta_{x_2})u$.

Apply the operator $1+\Delta_{x_1}$ to this equation, and

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=(1+\Delta_{x_1})(1+\Delta_{x_2})u.$$

By writing out the left member,

$$f(x_1,x_2)+f(x_1,x_2+\Delta_{x_2})-f(x_1,x_2)+f(x_1+\Delta x_1,x_2)-f(x_1 x_2)+f(x_1+\Delta x_1,x_2+\Delta x_2)$$

$$-f(x_1,x_2+\Delta x_2)-f(x_1+\Delta x_1,x_2)+f(x_1,x_2)$$

$$=f(x_1 x_2)+f(x_1+\Delta x_1,x_2+\Delta x_2)-f(x_1,x_2)=u+\Delta u.$$

The same process would show that by applying $(1+\Delta_{x_1})$ first, $(1+\Delta_{x_2})$ second, we would also get $u+\Delta u$. Hence would follow commutation of the two operators, so that

$$(1+\Delta_{x_1})(1+\Delta_{x_2})=(1+\Delta_{x_2})(1+\Delta_{x_1})=1+\Delta,$$

or if $u=f(x_1,x_2,\ldots\ldots x_n)$ we have proved that

$$(1+\Delta_{x_i})(1+\Delta_{x_k})=(1+\Delta_{x_k})(1+\Delta_{x_i})=1+\Delta_{x_i x_k},$$

i. e., any two of these operators are commutative. It follows $(n=2)$ that

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=u+\Delta_{x_1}u+\Delta_{x_2}u+\Delta_{x_2}\Delta_{x_1}u,$$

and since the ordinary addition is commutative, $\Delta_{x_1}\Delta_{x_2}u=\Delta_{x_2}\Delta_{x_1}u$, a familiar result.

Assume for $u=f(x_1,x_2\ldots\ldots x_n)$, that $u+\Delta u=(1+\Delta_{x_1})\ldots\ldots(1+\Delta_{x_n})u$.

Let $U=F(x_1,x_2\ldots\ldots x_{n+1})$.

Then by the assumption

$$U+\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})(1+\Delta_{x_2}\ldots\ldots(1+\Delta_{x_n})U.$$

Apply $(1+\Delta_{x_{n+1}})$ to both sides of this equation remembering that we have proved commutation of operators. We get,

$$U+\Delta_{x_1\ldots\ldots x_n}U+\Delta_{x_{n+1}}U+\Delta_{x_{n+1}}\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})\ldots\ldots(1+\Delta_{x_{n+1}})U.$$

By working out the left member, it becomes $U+\Delta U$, hence the next case takes the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true for $n=2$, it is true universally. The com-

mutation of any two operators brings with it the proof that $\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_r}u$ is equal to any other one of $r!$ orders in which the r operations might be brought about. The determinant form was suggested by the formula for the development of a determinant in terms of the elements of its principal diagonal, a formula which has the same limits and number of operators in the summation. It is easily shown by adding to each column the elements of the last, when it reduces to one term, its principal diagonal term.

2. The second formula is easily shown after it has been proved for two variables. Let $u=f(x_1, x_2)$. Now u may be composed linearly of a constant, a function of x_1 alone, a function of x_2 alone, and a function of (x_1, x_2). The last must always be present, though the others may be wanting. About the constant we are not here concerned. Neglecting it,

$$u=f_1(x_1)+f_2(x_2)+\phi(x_1, x_2).$$

$$d_{x_1}u=[f'(x_1)+D_{x_1}\phi]dx_1, \quad d_{x_2}u=[f_2'(x_2)+D_{x_2}\phi]dx_2,$$

$$d_{x_1}d_{x_2}u=D_{x_1}D_{x_2}\phi dx_1 dx_2. \quad \int d_{x_1}u=f_1(x_1)+\phi(x_1, x_2),$$

$$\int d_{x_2}u=f_2(x_2)+\phi(x_1, x_2). \quad \int\int d_{x_1}d_{x_2}u=\int\int D_{x_1}D_{x_2}\phi dx_1 dx_2=\phi(x_1, x_2).$$

Hence without a constant,

$$u=\int d_{x_1}u+\int d_{x_2}u-\int\int d_{x_1}d_{x_2}u.$$

i. e., we have here complete indefinite integral of du, and

$$(1-\int d)=(1-\int d_{x_1})(1-\int d_{x_2})u=0,$$

where it is evident that there is a commutation of operators, since the ordinary additions are commutative, and $d_{x_1}d_{x_2}u=d_{x_2}d_{x_1}u$.

Assume for $u=f(x_1, x_2\ldots\ldots x_n)$, that

$$(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=0. \quad \text{Let } U=F(x_1, x_2\ldots\ldots x_{n+1}).$$

Then $U=\int d_{x_1}U+\phi(x_2, x_3\ldots\ldots x_{n+1})$, where ϕ is an arbitrary function of all the variables but x_1. By the assumption

$$(1-\int d_{x_2})(1-\int d_{x_3})\ldots(1-\int d_{x_{n+1}})\phi=0, \text{ also } \phi=U-\int d_{x_1}U=(1-\int d_{x_1})U.$$

Apply the operator $(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_{n+1}})$ to both sides of this equation, remembering that commutation of operators has been proved. We get

$$\left(1-\int d_{x_2}\right)\dots\dots\left(1-\int d_{x_{n+1}}\right)\phi=\left(1-\int d_{x_1}\right)\dots\dots\left(1-\int d_{x_{n+1}}\right)U=0,$$

since the left member is zero ; also the last has the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true when $n=2$, it is universally true.

III. Applications.

1. An elegant application of the first development is its use in demonstrating that the total differential of a function is equal to the sum of its partial differentials. Let $u=f(x_1, x_2\dots\dots x_n)$. Then

$$\varDelta u=\varDelta_{x_1}u+\varDelta_{x_2}u+\dots\dots\varDelta_{x_n}u+\Sigma\varDelta_{x_i}\varDelta_{x_k}+\dots\dots\Sigma\varDelta_{x_i}\varDelta_{x_2}\dots\dots\varDelta_{x_n}u.$$

Multiply through by n, where n is a number which becomes indefinitely great as the principal increment becomes indefinitely small, but so that $\lim\limits_{\varDelta x_r=0}\left(n\varDelta_{x_r}u\right)=$a finite number, which is called by Hamilton, Serret, and J. M. Pierce, differential of u with respect to x_r, and may be denoted by $d_{x_r}u$. Consider $\lim\limits_{\substack{\varDelta x_r=0\\ \varDelta x_k=0}}\left(n\varDelta_{x_r}\varDelta_{x_k}u\right)$. This equals $\lim\limits_{\substack{\varDelta x_r=0\\ \varDelta x_k=0}}\left(\varDelta_{x_r}n\varDelta_{x_k}u\right)=\lim\limits_{\varDelta x_r\doteq 0}\varDelta_{x_r}\left(d_{x_k}\right)=0.$

A *fortiori*, zero will be the limit of any term of higher order. We have then at once by taking the limits of both members,

$$du=d_{x_1}+d_{x_2}u+\dots\dots d_{x_n}u.$$

2. The second formula gives us the complete solution of a differential equation, when it is a perfect differential. The solution is,

$$u=\int d_{x_1}u+\int d_{x_2}u+\dots\dots\int d_{x_n}u-\sum\int\int d_{x_i}d_{x_k}u+\sum\int\int\int d_{x_i}d_{x_k}d_{x_j}u\dots$$

$$+(-1)^{n-1}\int\int\dots\dots\int d_{x_1}d_{x_2}\dots\dots d_{x_n}u.$$

To this a constant may be added which is determined as usual by corresponding values of the variables and function. It may or may not be zero. When the function is such that farther differentiation of the first differentials will cut them down rapidly, this formula ought to be practically useful. When $n=2$, we can by this formula solve the problem of finding the orthogonal and isothermal curves to a given system, $u=c$, when u satisfies the equation $D_x^2 u+D_y^2 u=0$.